Causal Propagation of a Charged Spin 3/2 Field in an External Electromagnetic Background

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Abstract

We present a Lagrangian for a massive, charged spin 3/2 field in a constant external electromagnetic background, which correctly propagates only physical degrees of freedom inside the light cone. The Velo-Zwanziger acausality and other pathologies such as loss of hyperbolicity or the appearance of unphysical degrees of freedom are avoided by a judicious choice of non-minimal couplings. No additional fields or equations besides the spin 3/2 ones are needed to solve the problem.

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While completely explicit actions of *free* massive fields of spin arbitrarily larger than one − which propagate within the light cone the correct number of physical degrees of freedom − have been known since the 1970's [1], consistent actions for interacting fields have been much hard to construct. Indeed, even the conceptually simpler problem of describing high-spin particles in fixed external field backgrounds has proved itself fraught with difficulties. As already noticed by Fierz and Pauli seventy years ago [2], a Lagrangian formulation of interacting high-spin fields is essential even at the classical level, to avoid algebraic inconsistencies in the equations of motion. However, when the high-spin field is coupled to either external or dynamical fields, a Lagrangian formulation guarantees neither that no unphysical degrees of freedom start propagating, nor that the physical ones propagate only causally.

This pathology is particularly vexing for the seemingly simple case of charged, massive particles of spin 3/2. Their well-known free action was found in 1941 in [3], but it took many years before realizing that minimal coupling to external electromagnetic fields resulted in equations of motion which exhibited faster-than-light propagation of signals [4] (see also [5]). This lack of causality also shows up in higher spin fields, such as spin 2 [6].[2]

Massive, electrically charged states of spin 3/2 or higher do exist in QCD as resonances. Moreover, open string theory contains (infinitely) many charged, massive particles of spin higher than one. [3] Both string theory and QCD are to the best of our understanding consistent and causal, especially in the dynamical regime describing particles in fixed external electromagnetic fields. So, a natural question to ask is how the Velo-Zwanziger acausality problem is resolved, first of all in the simplest setting of them all: spin 3/2.

Possible Solutions

Various scenarios exist for rescuing causality.

Adding New Degrees of Freedom

One is that a single charged spin 3/2 field is inconsistent or non-causal when considered in isolation. It could happen that causality forces upon us the existence of other fields besides the spin 3/2 one. After all, we do know an example of consistently propagating charged spin 3/2 fields: $\mathcal{N} = 2$ extended "gauged" supergravity [9]. In $\mathcal{N} = 2$ theories, the gravitino can be charged under a $U(1)$ field (the graviphoton). Supersymmetry can be broken without introducing a cosmological constant [10, 11], resulting in a massive

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[2]The causality problem is a classical pathology; its quantum analog is that canonical commutators become ill-defined. The latter was noticed in [7] long before [4].

[3]Light spin 3/2 particles may also appear in Randall-Sundrum models [8].

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spin 3/2 field propagating in flat space. Causality in this case is due to gravitational back-reaction.

More specifically, as shown in [12], superluminal propagation of the mass m, charge e gravitino would occur in flat space when, in some Lorentz frame, the magnetic field $\mathbf{B}$ attains the critical value $|\mathbf{B}| = \sqrt{3}m^2/e$. In that frame, the energy density $T_{00} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)$ is always larger than $\frac{3}{2}m^4/e^2$. Since in this theory gravity is dynamical, the gravitational back-reaction induces a curvature in space-time, characterized by a length scale $L^{-2} = \mathcal{O}(3m^4/2M_{\mathrm{Pl}}^2e^2)$. But in $\mathcal{N} = 2$ theories, the graviphoton charge of the gravitino and the Planck mass M_{Pl} are related by $e = m/M_{\mathrm{Pl}}$; therefore, space-time is significantly curved already at the Compton wavelength scale of the gravitino $L = 1/m$. This is precisely the regime where the flat-space causality results of [4, 12] cease to apply. Indeed, ref. [13] extended the causality analysis done for pure supergravity in [14] to prove that $\mathcal{N} = 2$ supergravity is causal and hyperbolic when $m > \sqrt{\frac{2}{3}}eM_{\mathrm{Pl}}$. [4]

The main drawback of extended supergravity is that it cannot solve the causality problem of spin 3/2 fields unless the charge obeys the "Kaluza-Klein" relation $e = m/M_{\mathrm{Pl}}$. When $-$ as for electromagnetic interactions $-$ e is fixed ($e \approx 0.3$), the gravitational back-reaction of spin 3/2 particles much lighter than $\mathcal{O}(eM_{\mathrm{Pl}})$ is negligible, so they can still propagate superluminally.

Adding Non-Minimal Terms

A different solution to the causality problem may be to change the minimal spin 3/2 theory not by adding new dynamical degrees of freedom, but simply by adding non-minimal gauge invariant interactions. That this could be the right solution is strongly suggested by analogy with the only known example of a consistent model of high-spin particles of arbitrary charge, which propagates causally in an external electromagnetic field, constant but otherwise arbitrary. This is the Argyres-Nappi action [16]. It describes a single, massive spin 2 field, charged under a $U(1)$. Charge and mass are independent variables; in particular, a dynamical regime exists which decouples gravitational interactions, while keeping the $U(1)$ charge finite. The Argyres-Nappi action is highly non-minimal: it is quadratic in the charged spin 2 field but non-polynomial in the electromagnetic field strength $F_{\mu\nu}$. It was obtained from the equations of motion of charged open strings in a background with a nonzero, constant external field strength $F_{\mu\nu}$.

Even though derived within string theory, the reason why the Argyres-Nappi theory is causal and consistent is simple: After a straightforward redefinition of variables, its equations of motion enforce the standard transverse-tracelessness constraint on the spin 2 field $h_{\mu\nu}$. By substituting the constraint into the equations of motion, one obtains a good

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[4]Partial results on causality of $\mathcal{N} = 2$ and Kaluza-Klein supergravities can also be found in [15].

hyperbolic system, $\Box h_{\mu\nu} +$ lower derivative terms $= 0$, which manifestly propagates five degrees of freedom within the light cone.

It would be odd if what works with spin 2 does not work with spin 3/2, especially since the reason for causality in the Argyres-Nappi action is not due to exotic properties of string theory, but rather to a clever combination of non-minimal terms. So, even for spin 3/2, it makes sense to consider a general non-minimal Lagrangian of the form [5]

$$
\begin{aligned}
L &= -i\bar{\psi}_\mu A^{\mu\nu\rho}(F) D_\nu \psi_\rho - i\bar{\psi}_\mu B^{\mu\nu}(F) \psi_\nu, \\
A^{\mu\nu\rho}(F) &= \gamma^{\mu\nu\rho} + \mathcal{O}(F), \qquad B^{\mu\nu}(F) = m\gamma^{\mu\nu} + \mathcal{O}(F).
\end{aligned}
\tag{1}
$$

The non-minimal couplings $A^{\mu\nu\rho}(F), B^{\mu\nu}(F)$ are functions of the electromagnetic field strength $F_{\mu\nu}$, analytic near $F_{\mu\nu} = 0$. Their form will be specified later.

What We Cannot Expect to Find

Before analyzing further eq. (1), it is important to understand clearly what problem we must solve and which one we should not. Our aim is to find a Lagrangian that propagates within the light cone only four degrees of freedom − the four physical helicities of a spin 3/2 field − in an external electromagnetic background. Our method will work for constant backgrounds. While this is a drawback, it does take care of the original Velo-Zwanziger problem, which manifests itself already for constant backgrounds [4].

We do not want to find a Lagrangian that works for arbitrarily large values of the relativistic field invariants $F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\nu}\tilde{F}^{\mu\nu}$ [6]. The reason is that whenever these invariants become $\mathcal{O}(m^4/e^2)$, several instabilities appear, that make the very concept of a long-lived, propagating spin 3/2 field unphysical. One such instability is the Schwinger pair production [17], which becomes significant when $F_{\mu\nu}\tilde{F}^{\mu\nu} = 0$ and $F_{\mu\nu}F^{\mu\nu} \sim -m^4/e^2$. Another is the spin 3/2 analog [18] of the Nielsen-Olesen instabilities [19], which appear when $F_{\mu\nu}\tilde{F}^{\mu\nu} = 0$ and $F_{\mu\nu}F^{\mu\nu} \sim +m^4/e^2$. Though these instabilities are normally said to occur when either the electric field (Schwinger) or the magnetic field (Nielsen-Olesen) are $\mathcal{O}(m^2/e)$, it is important to realize that they only depend on relativistic invariant combinations of the field strength. These instabilities mean that whatever Lagrangian one may use to describe a spin 3/2 field in isolation, it will always be only an effective one, reliable only when energies are sufficiently small *and the relativistic field invariants are much smaller than* $\mathcal{O}(m^4/e^2)$. It is thus particularly telling that the Argyres-Nappi Lagrangian becomes ill-defined precisely when the relativistic field invariants reach their critical strength $\sim m^4/e^2$ [16].

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[5]Our conventions are as follows: the metric $\eta_{\mu\nu}$ is mostly plus, $\bar{\psi}_\mu = \psi_\mu^\dagger \gamma^0$, $\gamma^{\mu\dagger} = \eta^{\mu\mu}\gamma^\mu$, $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$. We always antisymmetrize with unit strength: $\gamma^{\mu_1\cdots\mu_n} = \frac{1}{n!}\gamma^{\mu_1}\gamma^{\mu_2}..\gamma^{\mu_n} +$ antisymmetrization.

[6]$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ with $\epsilon_{\mu\nu\rho\sigma}$ totally antisymmetric and normalized as $\epsilon_{0123} = +1$.

The Velo-Zwanziger problem is different in that it persists even at arbitrarily small values of the relativistic field invariants. Concretely, in the minimal model, the magnetic field $\mathbf{B}$ can reach its critical value $|\mathbf{B}| = \sqrt{\frac{3}{2}}m^2/e$ [7] in a frame where $|\mathbf{E}| = \sqrt{\frac{3}{2}}m^2/e - \epsilon$, with ϵ an arbitrarily small number. So, it is a real problem that occurs within the regime of validity of the effective theory.

This is the problem we need to solve: we must find a non-minimal Lagrangian that propagates causally the correct number of degrees of freedom whenever $|F_{\mu\nu}F^{\mu\nu}| \ll m^4/e^2$, $|F_{\mu\nu}\tilde{F}^{\mu\nu}| \ll m^4/e^2$.

We shall not worry if the Lagrangian fails whenever either of these invariants becomes $\mathcal{O}(m^4/e^2)$, since in any case *any* Lagrangian treating the electromagnetic background as fixed is meaningless, because it fails to take into account large effects due to *electromagnetic* back-reaction.

What is Not a Solution

Hermiticity of the Lagrangian in eq. (1) imposes some constraints on the coefficients $A^{\mu\nu\rho}(F), B^{\mu\nu}(F)$, namely

$$\gamma^0 (A^{\rho\nu\mu})^\dagger \gamma^0 = A^{\mu\nu\rho}, \qquad \gamma^0 (B^{\nu\mu})^\dagger \gamma^0 = -B^{\mu\nu}. \tag{2}$$

Moreover, unless $A^{\mu\nu\rho}$ and $B^{\mu\nu}$ are antisymmetric in their vector Lorentz indices, eq. (1) propagates additional degrees of freedom whenever $F_{\mu\nu} \neq 0$. These degrees of freedom are dangerous because they interact through relevant *and* irrelevant interactions with the electromagnetic field, but are absent at $F_{\mu\nu} = 0$, when the coefficients in the Lagrangian (1) are antisymmetric. So, their kinetic term is proportional to $|F_{\mu\nu}|$ and thus the strength of all their irrelevant interactions diverges in the weak field limit $|F_{\mu\nu}| \to 0$. The existence of these unwanted degrees of freedom makes the solution of the Velo-Zwanziger problem proposed in the appendix of [20] unacceptable, as pointed out in [12]. Yet, the idea that non-minimal interactions may cure the problem can be salvaged from that work, as we shall proceed to explain.

Constraints

What makes the Argyres-Nappi action work is that on an appropriately redefined spin 2 field it enforces the *same* constraint as the free action does, namely transverse-tracelessness [8]. Similarly, here we demand that our non-minimal action enforce the constraint

$$\gamma^\mu \psi_\mu = 0. \tag{3}$$

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[7] For minimal supergravity the critical value instead is $|\mathbf{B}| = \sqrt{3}m^2/e$.

[8] In the notation of ref. [16] the transverse-traceless field is $(HhH^*)_{\mu\nu}$.

We will present later a non-minimal action satisfying this requirement. It will turn out to have a canonical kinetic term $A^{\mu\nu\rho} = \gamma^{\mu\nu\rho}$. Before entering into the details of its construction, it is instructive to see why constraint (3) ensures at once that the equations of motion derived from Lagrangian (1) define a hyperbolic system that propagates causally four degrees of freedom.

When $A^{\mu\nu\rho} = \gamma^{\mu\nu\rho}$, the equations of motion are

$$R^\mu + B^{\mu\nu}\psi_\nu = 0, \qquad R^\mu \equiv \gamma^{\mu\nu\rho}D_\nu\psi_\rho, \tag{4}$$

while their gamma-trace is

$$2\gamma^{\mu\nu}D_\mu\psi_\nu + \dots = 0 \tag{5}$$

where the ellipsis stand for "mass" terms containing no derivatives. Using the identity

$$\gamma^{\mu\nu\rho} = \gamma^\mu\gamma^{\nu\rho} - \eta^{\mu\nu}\gamma^\rho + \eta^{\mu\rho}\gamma^\nu, \tag{6}$$

eq. (5) and the constraint (3), one can reduce equations of motion (4) to a standard, manifestly causal Dirac form:

$$\slashed{D}\psi_\mu + \text{ non-derivative terms} = 0. \tag{7}$$

Since $B^{\mu\nu}$ is antisymmetric in μ, ν, the 0-th component of the equations of motion, $R^0 = 0$ contains neither time derivatives nor ψ_0; thus, it enforces four constraints among the remaining fields ψ_i, $i = 1, 2, 3$. Constraint (3) then removes $\psi_0 = \gamma^0\gamma^i\psi_i$ leaving $3 \times 4 - 4 = 8$ physical variables i.e. four degrees of freedom (four coordinates and four conjugate momenta). A completely analogous way to prove the same result uses the obvious fact that consistent propagation of the constraint (3) and eq. (7) imply $D^\mu\psi_\mu = $ non-derivative terms; so, using eq. (7) to write $D_0\psi_0 = \gamma^0\gamma^i D_i\psi_0 + $ non-derivative terms, one gets from the above divergence the four additional constraints needed to reduce the number of degrees of freedom to four.

Construction of the Non-Minimal Action

Of course, the real question is whether a non-minimal action that gives eq. (3) exists. We prove that it does by explicitly constructing one.

Our ansatz for the non-minimal "Pauli" terms is

$$A_{\mu\nu\rho} = \gamma_{\mu\nu\rho}, \tag{8}$$
$$B_{\mu\nu} = m\gamma_{\mu\nu} + G^+_{\mu\nu} + \gamma^\rho T_{\rho[\mu}\gamma_{\nu]}, \tag{9}$$
$$G^+_{\mu\nu} \equiv G_{\mu\nu} + \tfrac{1}{2}\gamma_{\mu\nu\rho\sigma}G^{\rho\sigma} \tag{10}$$

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The Lorentz tensor $G_{\mu\nu}$ is antisymmetric ($G_{\mu\nu} = -G_{\nu\mu}$) and $\mathcal{O}(F)$, while the Lorentz tensor $T_{\mu\nu}$ is symmetric and traceless ($T_{\mu\nu} = T_{\nu\mu}$, $T^\mu_\mu = 0$) and $\mathcal{O}(F^2)$. Hermiticity of Lagrangian (1) implies that $T_{\mu\nu}$ is real and $G_{\mu\nu}$ is imaginary. Apart from these constraints, they are as-yet unspecified functions of the electromagnetic field strength $F_{\mu\nu}$.

As pointed out in ref [12], addition of $G^+_{\mu\nu}$ alone can never yield a causal theory, irrespective of its functional dependence on $F_{\mu\nu}$. It is crucial to notice that the term proportional to $T_{\mu\nu}$ instead, is structurally different from all those studied in [12] [9].

A few identities that will be crucial for our construction and follow from elementary manipulations of gamma-matrix algebra are (6) and

$$\gamma^\mu G^+_{\mu\nu} = \tfrac{1}{2}\gamma \cdot G\gamma_\nu, \qquad \gamma \cdot G \equiv \gamma_{\rho\sigma}G^{\rho\sigma}, \tag{11}$$

$$G^+_{\mu\nu} = -\tfrac{1}{4}\gamma^\rho\gamma \cdot G\gamma_{\rho\mu\nu} \tag{12}$$

$$\gamma^\rho D_{[\rho}\psi_{\mu]} = \tfrac{1}{2}R_\mu - \tfrac{1}{4}\gamma_\mu\gamma^\rho R_\rho, \qquad \gamma^{\mu\nu}D_\mu\psi_\nu = \tfrac{1}{2}\gamma^\rho R_\rho. \tag{13}$$

Either direct calculation or simple considerations of representation theory of the Lorentz group lead to another important identity [10]

$$G_{\mu\rho}\tilde{G}^{\rho\nu} = -\tfrac{1}{4}\delta^\nu_\mu G_{\rho\sigma}\tilde{G}^{\rho\sigma}. \tag{14}$$

Thanks to these identities, the gamma-trace of the equations of motion (4) is

$$2\gamma^{\mu\nu}D_\mu\psi_\nu + T^{\mu\nu}\gamma_\mu\psi_\nu = [-3m + \mathcal{O}(F)]\,\gamma \cdot \psi. \tag{15}$$

The term multiplying $\gamma \cdot \psi \equiv \gamma^\mu\psi_\mu$ on the right-hand side of this equation is a 4×4 matrix containing no derivatives, thus built only out of gamma matrices, $G_{\mu\nu}$, and $T_{\mu\nu}$. The split into the constant term $-3m$ and higher powers of the electromagnetic field follows simply from our ansatz, $G_{\mu\nu} = \mathcal{O}(F)$, $T_{\mu\nu} = \mathcal{O}(F^2)$.

Next we take the divergence of equations of motion (4). Since the covariant derivative D_μ obeys $[D_\mu, D_\nu] = ieF_{\mu\nu}$ we have

$$D_\mu R^\mu = -ieF^{\mu\nu}\gamma_\mu\psi_\nu + \frac{ie}{2}\gamma \cdot F\gamma \cdot \psi. \tag{16}$$

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[9]Appendix B of [12] proves that non-minimal terms of the generic form $B_{\mu\nu} = iW_{\mu\nu} + \gamma^5 X_{\mu\nu} + \gamma_{\mu\nu}Y + i\gamma^5\gamma_{\mu\nu}Z$ still allow for superluminal propagation, even when the coefficients W, X, Y, Z are arbitrary functions of $F_{\mu\nu}$. To the best of our knowledge, the last term in our eq. (9) has never been considered before.

[10]$G_{\mu\nu}$ decomposes into irreps of $SL(2,C)$ as $(1,0) + (0,1)$ while $\tilde{G}_{\mu\nu}$ decomposes as $(1,0) - (0,1)$. The most general tensor product of the two decomposes as $(2,0) + (1,0) + (0,1) + (0,0)$, but since its antisymmetric part vanishes on self-dual or anti self-dual backgrounds, it cannot contain either $(1,0)$ or $(0,1)$. On the other hand, the same tensor product can only contain representations appearing in the tensor product $(\tfrac{1}{2}, \tfrac{1}{2}) \times (\tfrac{1}{2}, \tfrac{1}{2}) = (1,1) + (1,0) + (0,1) + (0,0)$. The only common element is $(0,0)$.

By using eq. (15), identities (11-13) plus the vanishing of $\gamma^\mu \gamma_{\alpha\beta} \gamma_\mu$ and $\gamma^\mu T_{\mu\nu} \gamma^\nu$ [11] we re-write the divergence as

$$-ie\gamma_\mu F^{\mu\nu}\psi_\nu - \tfrac{1}{2}m\gamma_\mu T^{\mu\nu}\psi_\nu - \tfrac{1}{4}\gamma^\mu \gamma \cdot G\left[m\eta_{\mu\nu} - G^+_{\mu\nu} + T_{\mu\nu}\right]\psi^\nu$$
$$-\tfrac{1}{2}\gamma_\rho T^{\rho\mu}\left[m\eta_{\mu\nu} - G^+_{\mu\nu} + T_{\mu\nu}\right]\psi^\nu = \left[\tfrac{3}{2}m^2 + \mathcal{O}(F)\right]\gamma\cdot\psi. \tag{17}$$

In this equation we use again identities (11-13) as well as identity (14) to simplify the term quadratic in G; we obtain

$$\gamma^\mu\left(-ieF_{\mu\nu} + mG_{\mu\nu} + \tfrac{1}{2}T_\mu^\rho G^+_{\rho\nu} - \tfrac{1}{2}G^+_{\mu\rho}T^\rho_\nu + G_{\mu\rho}T^\rho_\nu\right)\psi^\nu$$
$$-\gamma^\mu\left(G_\mu{}^\rho G_{\rho\nu} + mT_{\mu\nu} + \tfrac{1}{2}T_\mu^\rho T_{\rho\nu}\right)\psi^\nu = \left[\tfrac{3}{2}m^2 + \mathcal{O}(F)\right]\gamma\cdot\psi. \tag{18}$$

Two conditions must be met to enforce the standard constraint $\gamma\cdot\psi = 0$. The first is that the left-hand side of eq. (18) must either vanish or be proportional to $\mathcal{O}(F)\gamma\cdot\psi$; the second is that the matrix $\left[\tfrac{3}{2}m^2 + \mathcal{O}(F)\right]$ is invertible.

The hard one is the first.

To satisfy it, we first of all set

$$T^\nu_\mu = A\left(G_{\mu\rho}G^{\rho\nu} - \tfrac{1}{4}G_{\sigma\rho}G^{\rho\sigma}\delta^\nu_\mu\right), \tag{19}$$

where A is a constant. This choice renders the term $T^\rho_\mu T_{\rho\nu} = \mathcal{O}(F^4)\gamma\cdot\psi$ and also makes the term inside the first parenthesis in eq. (18) antisymmetric in μ,ν. Both properties follow from eq. (14), which gives the identities ($\operatorname{Tr}H \equiv H^\mu_\mu$)

$$G^+_{\mu\rho}G^{-\,\rho\nu} = G^-_{\mu\rho}G^{+\,\rho\nu} = 2\left[G_{\mu\rho}G^{\rho\nu} - \tfrac{1}{4}\operatorname{Tr}(G^2)\delta^\nu_\mu\right] \tag{20}$$
$$G^\pm_{\mu\rho}G^{\pm\,\rho\nu} = \tfrac{1}{2}\left[\operatorname{Tr}(G^2) \pm i\gamma^5\operatorname{Tr}(G\tilde{G})\right]\delta^\nu_\mu. \tag{21}$$

Now the two terms in parentheses in eq. (18) must separately either vanish or be proportional to $\gamma\cdot\psi$, since the first is antisymmetric in μ,ν while the second is symmetric.

The choice $A = -m^{-1}$ makes the whole symmetric term in (18) equal to $\mathcal{O}(F^2)\gamma\cdot\psi$. On the other hand, the antisymmetric term vanishes if $G_{\mu\nu}$ satisfies the following implicit equation:

$$G_{\mu\nu} = +\frac{ie}{m}F_{\mu\nu} + \frac{1}{4m^2}\operatorname{Tr}(G^2)G_{\mu\nu} - \frac{1}{4m^2}\operatorname{Tr}(G\tilde{G})\tilde{G}_{\mu\nu}. \tag{22}$$

This can be solved by power series as long as the relativistic field invariants $F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\nu}\tilde{F}^{\mu\nu}$ have magnitudes that are small compared to m^4/e^2 [12]. This is the crucial feature we need, namely a theory that only breaks down for large *invariants*, but that is well-behaved when they are small, even when some field strength component becomes $\mathcal{O}(m^2/e)$.

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[11] The first is a standard gamma-matrix identity, while the second follows from tracelessness of $T_{\mu\nu}$.

[12] (Semi)-explicitly, $G_{\mu\nu} = aF_{\mu\nu} + b\tilde{F}_{\mu\nu}$, with a,b analytic functions of the relativistic field invariants. They obey $a = ie/m + \mathcal{O}[\operatorname{Tr}(F^2),\operatorname{Tr}(F\tilde{F})]$, $b = \mathcal{O}[\operatorname{Tr}(F^2),\operatorname{Tr}(F\tilde{F})]$ for $|\operatorname{Tr}(F^2)|, |\operatorname{Tr}(F\tilde{F})| \ll m^4/e^2$.

Eqs. (19,22) make the constraint (18) take the desired form

$$\left[\tfrac{3}{2}m^2 + \mathcal{O}(F)\right]\gamma \cdot \psi = 0. \tag{23}$$

The proportionality matrix multiplying $\gamma \cdot \psi$ contains only gamma matrices and powers of $(e/m^2)F_{\mu\nu}$ with dimensionless coefficients; therefore, Lorentz invariance implies that its determinant can only be a function of relativistic field invariants, hence invertible when $|F_{\mu\nu}F^{\mu\nu}|, |F_{\mu\nu}\tilde{F}^{\mu\nu}| \ll m^4/e^2$. So the second condition is met precisely when an effective Lagrangian description is supposed to make sense.

The redefinition $G_{\mu\nu} = imX_{\mu\nu}$ and a straightforward computation give the explicit form of the matrix; the constraint equation then becomes

$$\left\{48 - [\mathrm{Tr}\,(X^2)]^2 - [\mathrm{Tr}\,(X\tilde{X})]^2\right\}\gamma \cdot \psi = 0, \tag{24}$$

which manifestly depends only on relativistic field invariants.

Summary

The construction presented in this paper answers a question that in various guises remained unanswered for many decades, namely: does a consistent, causal Lagrangian describing a single massive, charged particle of spin larger than one in interaction with the electromagnetic field exist?

The answer for spin $3/2$ is yes, at least for constant external fields. This is a major achievement in itself, since constant fields are exactly those that cause the Velo-Zwanziger acausality [4] and the Johnson-Sudarshan problem [7].

The crucial property of our construction is that the standard gamma-tracelessness constraint $\gamma \cdot \psi = 0$ is enforced *exactly*. Enforcing it only up to a finite order in an expansion in powers of the field strength would not suffice. To see this, we may try to substitute $G_{\mu\nu} = i(e/m)F_{\mu\nu}, T_{\mu\nu} = 0$ in our non-minimal ansatz eq. (9). This choice satisfies constraint (3) up to $\mathcal{O}(F^2)$:

$$\left[\tfrac{3}{2}m^2 + \mathcal{O}(F)\right]\gamma \cdot \psi = \frac{e^2}{m^2}\gamma^{\mu}F_{\mu\rho}F^{\rho\nu}\psi_{\nu}. \tag{25}$$

As shown in [12], superluminal propagation of signals occurs when $\psi_0 \neq 0$, $\psi_i = 0$ solve this equation. Contrary to, say, constraint eq. (24), eq. (25) depends on quantities, such as the electromagnetic stress energy tensor, that can be large even when the relativistic field invariants are small. This property allows $\psi_0 \neq 0$, $\psi_i = 0$ to be a solution even when $|\mathrm{Tr}\,(F^2)|, |\mathrm{Tr}\,(F\tilde{F})| \ll m^4/e^2$, as it can be easily proven by direct computation [12].

It is also illuminating that the solution involves no extra degrees of freedom and that carefully chosen parity-preserving non-minimal terms suffice. It is an amusing and perhaps

deep fact that the non-minimal couplings also give a gyromagnetic factor $g = 2 -$ the same value needed to improve the high-energy behavior of "Compton" forward scattering amplitudes, and the one given by open string theory [20].

It is finally worth noticing that the non-minimal terms required by causality in our admittedly non-unique Lagrangian lower the intrinsic UV cutoff of the theory, from its theoretical maximum $\Lambda \sim e^{-1/2}m$ [21], to $\Lambda \sim e^{-1/3}m$. If this property were to extend to the most general causal Lagrangian of charged spin $3/2$ fields it would offer a powerful tool to establish stronger, model independent limits on the UV cutoff of such theories.

Acknowledgments

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